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Integrals of motion and semi-regular Lepagean forms in higher-order mechanics

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Abstract. In this paper a relation between a canonical set of integrals of motion and the Lepagean (fundamental) differential form θ in higher-order mechanics is studied. The canonical set of integrals of motion is introduced as a system of functions of coordinates, time and higher-order velocities of the given mechanical system, satisfying certain axioms of functional independence, completeness and canonical adjointness. It is shown that there exists a correspondence between canonical sets of integrals of motion and semi-regular Lepagean forms. A connection of invariance transformations of the form $d\theta$ with integrals of motion is studied, and a generalisation of the local Liouville theorem on the integrals of motion in involution is given.

1. Introduction

There are a few approaches to the problem of a generalisation of the well known Poincaré–Cartan form (Goldschmidt and Sternberg 1973) to higher-order calculus of variations, all of them leading to the same result in higher-order mechanics. For instance Krupka (1973, 1982) introduced the concept of a Lepagean form, related to a certain 'horizontalisation' of forms, Aldaya and Azcárraga (1978, 1980) applied the Lagrange multipliers, and Shadwick (1982) adapted a procedure of working modulo the so-called contact forms.

The purpose of this paper is to study the relation and the conditions of local equivalence between the motion laws and the conservation laws in higher-order mechanics. We introduce the canonical set of integrals of motion related to a Lepagean form; the basic tool applied is the Darboux theorem on the canonical form of an exterior two-form (see e.g. Sternberg 1964). The concept of an invariant transformation of a Lepagean form is used analogously as in the first-order theory (see e.g. Sarlet and Cantrijn 1981). We derive a relation for the calculation of Lagrange functions from the canonical set of integrals of motion and we apply this relation to the inverse problem of the calculus of variations. Finally, we prove a generalisation of the classical Liouville theorem on the integrability of equations of motion (see e.g. Arnold 1979) in the local case to higher-order mechanics and we introduce explicit relations for the corresponding quadratures.

For a kinematic description of a mechanical system we use a fibred manifold $\pi: W \to I$ and its jet prolongations. The basis I is a one-dimensional time interval and $W = I \times X$, where X is an n-dimensional configuration manifold of the mechanical system. The time evolution of the mechanical system then corresponds to a certain section γ of the fibred manifold π. Further on, we use the following notations: j^rW is an r-jet prolongation of W, $j^r\gamma$ is an r-jet prolongation of γ, $\pi_{r,0}$ a natural projection of j^rW to W, (V, ψ), $\psi = (t, q^\sigma)$ a local coordinate system on W, (V_r, ψ_r), $\psi_r = (t, q_0^\sigma, q_1^\sigma, q_2^\sigma, \ldots, q_r^\sigma)$ a local coordinate system on j^rW associated with (V, ψ). We also use for the coordinates q_k^σ the traditional notation $q_0^\sigma = q^\sigma$, $q_1^\sigma = \dot{q}^\sigma$, $q_2^\sigma = \ddot{q}^\sigma$, etc. For Greek indices we use everywhere the standard summation convention, and summation is always done within maximum limits. Further, we use the following differential-geometrical operations: d is the exterior derivative, i_ξ the contraction of a form by a vector field ξ, ∂_ξ the Lie derivative with respect to the vector field ξ, h the horizontalisation of forms on j^rW, $\mathrm{d}/\mathrm{d}t$ the total derivative, f^* the pull-back of a form by a mapping f, f_* the tangent mapping of a mapping f. The manifold X and all mappings which are used in this paper are supposed to be infinitely differentiable.

2. Lepagean forms

Definition 1. We say that a one-form θ on an open set $U \subset j^rW$ is a *Lepagean* form if for each vector field ξ on U such that $(\pi_{r,0})_*\xi = 0$ it holds that

$$h(i_\xi\, \mathrm{d}\theta) = 0. \tag{2.1}$$

The mapping h (horizontalisation) used in definition 1 is defined in (Krupka 1973, 1982) as a mapping assigning to each k-form ρ on j^rW a π_r-horizontal k-form $h(\rho)$ on $j^{r+1}W$ such that $j^r\gamma^*\rho = j^{r+1}\gamma^*h(\rho)$ for each section γ of π; h is linear over the ring of functions. In mechanics, where the base of π is one-dimensional, we have for any function F on j^rW, $h(\mathrm{d}F) = (\mathrm{d}F/\mathrm{d}t)\,\mathrm{d}t$. This relation is often used in practical calculations.

Definition 2. Let θ be a Lepagean form defined on an open set $U \subset j^rW$. We say that a local section γ of the fibred manifold π is a *critical section* with respect to θ if it holds that:

(i) the prolongation $j^r\gamma$ of the section γ lies in U;
(ii) for each vector field ξ on U

$$j^r\gamma^* i_\xi\, \mathrm{d}\theta = 0. \tag{2.2}$$

Definition 2 expresses the physical meaning of the Lepagean form. Each Lepagean form θ determines a certain dynamics of the mechanical system. Within the frame of this dynamics only motions of the mechanical system are possible which correspond to the critical sections of the fibred manifold π (with respect to θ).

Definition 3. We say that the one-form θ on an open set $U \subset j^rW$ is *semi-regular* if there exists a positive integer s such that at each point of U

$$\mathrm{rank}(\mathrm{d}\theta) = 2s. \tag{2.3}$$

A regular Lepagean form is a special case of a semi-regular Lepagean form. Such a case occurs if in definition 3, $s = \frac{1}{2}n(r+1)$. In first-order mechanics, where $r = 1$, a Lepagean form is regular if and only if $s = n$.

3. A canonical set of the integrals of motion

Definition 4. We say that the functions $P_1, P_2, \ldots, P_s, Q^1, Q^2, \ldots, Q^s$ on an open set $U \subset j^rW$ form a *canonical set* Φ of integrals of motion on U if:

(i) the one-forms $dP_1, dP_2, \ldots, dP_s, dQ^1, dQ^2, \ldots, dQ^s$ are linearly independent at every point of U;

(ii) for each vector field ξ on U such that $(\pi_{r,0})_*\xi = 0$

$$i_\xi[(dQ^\alpha/dt)dP_\alpha - (dP_\alpha/dt)dQ^\alpha] = 0. \tag{3.1}$$

The condition (ii) of definition 4 ensures that the integrals of the set Φ are pairwise *canonically associated*; however, it also guarantees a certain *completeness* of the set. As the summation in (3.1) ranges from 1 to s, a subset of Φ or an extension will in general not form a canonical set.

Similarly to the Lepagean form θ, the canonical set Φ of integrals of motion also determines a certain dynamics of the mechanical system expressed by means of a critical section of the fibred manifold π.

Definition 5. Let Φ be the canonical set of integrals of motion P_α, Q^α $(\alpha = 1, 2, \ldots, s)$ on an open set $U \subset j^rW$. We say that a local section γ of a fibred manifold π is a *critical section* with respect to Φ if:

(i) the prolongation $j^r\gamma$ of the section γ lies in U;

(ii) for each $\alpha = 1, 2, \ldots, s$

$$j^r\gamma^*\, dP_\alpha = 0, \qquad j^r\gamma^*\, dQ^\alpha = 0. \tag{3.2}$$

Within the frame of the dynamics determined by the canonical set Φ of integrals of motion, only motions of the mechanical system are possible which correspond to the critical sections γ of the fibred manifold π. Equations (3.2) then represent the conservation laws of integrals of motion of the canonical set Φ.

4. Equivalence of the Lepagean form and the canonical set of integrals of motion

Definition 6. We say that the Lepagean form θ on an open set $U \subset j^rW$ and the canonical set Φ of integrals of motion P_α, Q^α $(\alpha = 1, 2, \ldots, s)$ on U are *equivalent* if the conditions (2.2) and (3.2) for critical sections γ of the fibred manifold π are equivalent on a set U.

Definition 6 is expressing, in other words, that ϕ and θ are equivalent if and only if conditions (2.2) and (3.2) have the same solutions γ, i.e. when Φ and θ determine the same dynamics of the mechanical system. The following theorem is concerned with the conditions of existence of a Lepagean form θ and an equivalent canonical set Φ of integrals of motion.

Theorem 1.

(i) For every semi-regular Lepagean form θ on an open set $U \subset j^rW$ there exists an equivalent canonical set Φ of integrals of motion on the neighbourhood of every point of the set U.

(ii) For every canonical set Φ of integrals of motion on an open set $U \subset j^r W$ there exists an equivalent semi-regular Lepagean form θ on U.

Proof. (i) The form $d\theta$ is a closed two-form on a $(1+n+rn)$-dimensional manifold U and it has, according to definition 3, a constant rank equal to $2s$. The assumptions of the Darboux theorem are then satisfied (e.g. Sternberg 1964) and therefore it is possible in a neighbourhood of each point to introduce coordinates

$$P_1, P_2, \ldots, P_s, Q^1, Q^2, \ldots, Q^s, x^1, x^2, \ldots, x^{1+n+rn-2s} \tag{4.1}$$

on which there holds

$$d\theta = dP_\alpha \wedge dQ^\alpha. \tag{4.2}$$

As for each vector field

$$\xi = \xi_\alpha \partial/\partial P_\alpha + \xi^\alpha \partial/\partial Q^\alpha + \zeta^\gamma \partial/\partial x^\gamma \tag{4.3}$$

it holds on U that

$$h(i_\xi\, d\theta) = [(i_\xi\, dP_\alpha)\, dQ^\alpha/dt - (i_\xi\, dQ^\alpha)\, dP_\alpha/dt]\, dt \tag{4.4}$$

the conditions of definition 4 according to definition 1 are satisfied, so that the set Φ formed by the coordinates $P_1, P_2, \ldots, P_s, Q^1, Q^2, \ldots, Q^s$ is a canonical set of the integrals of motion. Since the vector field ξ is arbitrary, the relation

$$j^r\gamma^* i_\xi\, d\theta = j^r\gamma^*(\xi_\alpha\, dQ^\alpha - \xi^\alpha\, dP_\alpha) \tag{4.5}$$

implies that Φ and θ are equivalent in the sense of definition 6.

(ii) By means of the canonical set Φ of integrals of motion $P_1, P_2, \ldots, P_s, Q^1, Q^2, \ldots, Q^s$ let us construct a one-form on U

$$\theta = P_\alpha\, dQ^\alpha + dS \tag{4.6}$$

where S is an arbitrary function on U. This form is a Lepagean form as is evident from (2.1), (3.1), (4.4). According to the relation

$$d\theta \wedge d\theta \wedge \ldots \wedge d\theta = s!\, dP_1 \wedge dQ^1 \wedge dP_2 \wedge dQ^2 \wedge \ldots \wedge dP_s \wedge dQ^s \tag{4.7}$$

where on the left-hand side there are s factors $d\theta$, and according to (i) of definition 4, the form (4.6) is also semi-regular in the sense of definition 3. On a neighbourhood of each point of the set U we can introduce coordinates (4.1) and therefore here Φ and θ are, according to (2.2), (3.2), (4.5) equivalent in the sense of definition 6. In this case, however, Φ and θ are equivalent over the whole set U and the proof of theorem 1 is finished.

It follows from the proof that the correspondence of Φ and θ is not one-to-one. For a given canonical set Φ there can exist more equivalent Lepagean forms θ, which differ by the function S in (4.6). Analogously, for a fixed chosen Lepagean form θ there can exist more equivalent canonical sets Φ of integrals of motion, because the transformation of the form $d\theta$ to the canonical form (4.2) is not unique.

5. The Lagrange function associated with the Lepagean form

The relation between the Lepagean forms and the corresponding Lagrange functions is studied by Krupka (1982). The most important results and procedures of that paper are summarised for higher-order mechanics in the following assertion and its proof.

Theorem 2. Let (V, ψ), $\psi = (t, q^\sigma)$ be a local coordinate system on W. Then to each Lepagean form θ on V_r is assigned a *Lagrange function* L on V_{r+1} by the relation

$$h(\theta) = L \, \mathrm{d}t. \tag{5.1}$$

The Euler–Lagrange equations for critical sections γ of the fibred manifold π

$$j^{r+1}\gamma^* \varepsilon_\sigma = 0, \tag{5.2}$$

where

$$\varepsilon_\sigma = \sum_{k=0}^{r+1} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial q^\sigma_k}\right), \tag{5.3}$$

are equivalent to the condition (2.2).

Proof. The Lepagean form θ has in the coordinate system (V_r, ψ_r) the form

$$\theta = M \, \mathrm{d}t + \sum_{k=0}^{r} L^k_\sigma \, \mathrm{d}q^\sigma_k. \tag{5.4}$$

Therefore

$$h(\theta) = \left(M + \sum_{k=0}^{r} L^k_\sigma q^\sigma_{k+1}\right) \mathrm{d}t. \tag{5.5}$$

According to (5.1) it holds that

$$M = L - \sum_{k=0}^{r} L^k_\sigma q^\sigma_{k+1} \tag{5.6}$$

so that

$$\theta = L \, \mathrm{d}t + \sum_{k=0}^{r} L^k_\sigma (\mathrm{d}q^\sigma_k - q^\sigma_{k+1} \, \mathrm{d}t). \tag{5.7}$$

As the form (5.4) is defined on V_r, neither M nor L^k_σ can depend on q^σ_{r+1}. By differentiating (5.6) with respect to q^σ_{r+1}

$$L^r_\sigma = \partial L / \partial q^\sigma_{r+1}. \tag{5.8}$$

Now, let us consider a vector field

$$\xi = \zeta \frac{\partial}{\partial t} + \sum_{k=0}^{r} \xi^\sigma_k \frac{\partial}{\partial q^\sigma_k}. \tag{5.9}$$

The condition $(\pi_{r,0})_* \xi = 0$ implies $\zeta = 0$, $\xi^\sigma_0 = 0$ and the other components can be arbitrary. In such a case we get

$$h(i_\xi \, \mathrm{d}\theta) = \sum_{k=1}^{r} \left(\frac{\partial L}{\partial q^\sigma_k} - \frac{\mathrm{d}L^k_\sigma}{\mathrm{d}t} - L^{k-1}_\sigma\right) \xi^\sigma_k \, \mathrm{d}t \tag{5.10}$$

and from here by means of definition 1 the recurrence relation

$$L_\sigma^{k-1} = \partial L/\partial q_k^\sigma - \mathrm{d}L_\sigma^k/\mathrm{d}t, \tag{5.11}$$

where $k = 1, 2, \ldots, r$. From the condition (5.8) by means of (5.11) it follows that

$$L_\sigma^k = \sum_{l=0}^{r-k} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\partial L}{\partial q_{k+l+1}^\sigma}\right). \tag{5.12}$$

Now, consider the case when the vector field (5.9) is quite an arbitrary one. From (5.3), (5.7) and (5.12) it follows that

$$j^r\gamma^* i_\xi \,\mathrm{d}\theta = j^{r+1}\gamma^* \varepsilon_\sigma(\xi_0^\sigma - \dot{q}^\sigma\zeta)\,\mathrm{d}t \tag{5.13}$$

from which the equivalence of (2.2) and (5.2) is apparent. This completes the proof.

It is obvious that for each Lagrange function L on V_{r+1} the Lepagean form can be found by means of (5.7), (5.12)

$$\theta = L\,\mathrm{d}t + \sum_{k=0}^{r}\sum_{l=0}^{r-k} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\partial L}{\partial q_{k+l+1}^\sigma}\right)(\mathrm{d}q_k^\sigma - q_{k+1}^\sigma\,\mathrm{d}t) \tag{5.14}$$

which, however, need not be, in general, defined on V_r. By more detailed considerations one can find the conditions which must be satisfied by the Lagrange function L on V_{r+1} so that the corresponding Lepagean form (5.14) might be defined on V_r.

Theorems 1 and 2 can be used to solve certain modifications of the inverse variational problem. If we know the integrals of motion $P_1, P_2, \ldots, P_s, Q^1, Q^2, \ldots, Q^s$ of some canonical set Φ, we can determine the corresponding Lagrange function by means of (4.6), (5.1)

$$L = P_\alpha\,\mathrm{d}Q^\alpha/\mathrm{d}t + \mathrm{d}S/\mathrm{d}t \tag{5.15}$$

associated with the Lepagean form θ equivalent to Φ. The arbitrary function S here has the meaning of the corresponding action of the mechanical system, calculated along a critical section. The inverse problem to determine a canonical set of integrals of motion from a Lagrange function L is a special case of the well known problem of finding a normal form (4.2) of an arbitrary closed two-form of constant rank. In §§ 6 and 7, we apply the general theory of the normal form to constructing a canonical set of integrals of motion.

6. Invariant transformations

Invariant transformations of the form $\mathrm{d}\theta$ are used in first-order mechanics to search for integrals of motion. It is shown in the following theorem that the invariant transformations can be analogously used even in higher-order mechanics and at the same time it joins them to the whole canonical set of integrals of motion.

Theorem 3. Let $U \subset j^rW$ be an open set, $\dim U = 1 + n + rn$, θ a semi-regular Lepagean form on U, Φ a canonical set of integrals of motion P_α, Q^α ($\alpha = 1, 2, \ldots, s$) on U and equivalent to θ. Then the following hold.

(i) To each vector field ξ on U, generating a one-parameter group of invariant transformations of the form $d\theta$, there is assigned locally a function F of the elements of the canonical set of integrals of motion P_α, Q^α, by the relation

$$i_\xi \, d\theta = dF. \tag{6.1}$$

The function F is determined explicitly up to an arbitrary additive constant.

(ii) To each function F of the elements P_α, Q^α of the canonical set Φ of integrals of motion on U there is by relation (6.1) locally assigned a generator ξ of a one-parameter group of invariant transformations of the form $d\theta$. The vector field ξ is determined explicitly up to an arbitrary additive vector field η, which belongs to the $(1+n+rn-2s)$-dimensional distribution of the solutions of the equation

$$i_\eta \, d\theta = 0. \tag{6.2}$$

Proof.

(i) As ξ is a generator of a one-parameter group of invariant transformations of the form $d\theta$, it holds that

$$\partial_\xi \, d\theta = 0. \tag{6.3}$$

Since

$$\partial_\xi \, d\theta = d i_\xi \, d\theta, \tag{6.4}$$

according to the Poincaré lemma there exists, locally, a function F so that (6.1) holds. The function F is determined up to an arbitrary additive constant. In the coordinate system (4.1) let the vector field ξ be expressed by

$$\xi = \xi_\alpha \partial/\partial P_\alpha + \xi^\alpha \partial/\partial Q^\alpha + \eta^\gamma \partial/\partial x^\gamma. \tag{6.5}$$

According to (4.2) and (6.1) it holds that

$$\xi_\alpha dQ^\alpha - \xi^\alpha \, dP_\alpha = \frac{\partial F}{\partial P_\alpha} dP_\alpha + \frac{\partial F}{\partial Q^\alpha} dQ^\alpha + \frac{\partial F}{\partial x^\gamma} dx^\gamma. \tag{6.6}$$

From here

$$\partial F/\partial x^\gamma = 0 \tag{6.7}$$

so that the function F depends on the variables P_α, Q^α of the canonical set Φ only.

(ii) For each function F of variables P_α, Q^α it follows from (6.6) that in the coordinate system (4.1)

$$\xi_\alpha = \partial F/\partial Q^\alpha, \qquad \xi^\alpha = -\partial F/\partial P_\alpha, \tag{6.8}$$

so that according to (6.5) and (6.8), the vector field

$$\xi = \frac{\partial F}{\partial Q^\alpha} \frac{\partial}{\partial P_\alpha} - \frac{\partial F}{\partial P_\alpha} \frac{\partial}{\partial Q^\alpha} + \eta^\gamma \frac{\partial}{\partial x^\gamma}, \tag{6.9}$$

with arbitrary components η^γ, is a solution of (6.1) and so according to (6.3) and (6.4) it generates a one-parameter group of invariant transformations of the form $d\theta$. Thereby, according to (4.2), the vector field

$$\eta = \eta^\gamma \partial/\partial x^\gamma \tag{6.10}$$

is a general solution of (6.2).

7. The Liouville theorem

It is not always possible to determine all $2s$ integrals of motion of the canonical set Φ just by means of invariant transformations of the form $d\theta$ alone. In the regular case of first-order mechanics it is, however, sufficient according to the classical Liouville theorem to determine s integrals in involution and the remaining integrals can then be determined by means of quadratures. The following definition and theorem generalise this process in the local case even to higher-order mechanics.

Definition 7. We say that the functions $P_1, P_2, \ldots, P_s$ on an open set $U \subset j^r W$ are *independent integrals of motion in involution* with respect to the Lepagean form θ on U, if it holds that:

(i) one-forms $dP_1, dP_2, \ldots, dP_s$ are at each point of the set U linearly independent;

(ii) there exist vector fields $\xi_1, \xi_2, \ldots, \xi_s$ on U such that for all $\alpha, \beta = 1, 2, \ldots, s$

$$i_{\xi_\alpha} d\theta = dP_\alpha, \tag{7.1}$$

$$i_{\xi_\alpha} dP_\beta = 0. \tag{7.2}$$

Theorem 4. Let $P_1, P_2, \ldots, P_s$ be independent integrals of motion on an open set $U \subset j^r W$, which are in involution with respect to the semi-regular Lepagean form θ on U such that $\operatorname{rank}(d\theta) = 2s$. Then in the neighbourhood of each point of the set U there exist functions $Q^1, Q^2, \ldots, Q^s$ which, together with integrals $P_1, P_2, \ldots, P_s$, form a canonical set of integrals of motion, equivalent to the Lepagean form θ.

Proof. Let us consider a point $x_0 \in U$. According to (i) of definition 7 it is possible to choose coordinates in a neighbourhood of the point x_0

$$P_1, P_2, \ldots, P_s, x^1, x^2, \ldots, x^{1+n+rn-s}. \tag{7.3}$$

As the integrals P_α are according to (i) of theorem 3 determined up to additive constants, we can suppose without loss of generality that all coordinates (7.3) of the point $x_0 \in U$ are zero.

The coordinate expression of the solution ξ_α of (7.1) cannot contain $\partial/\partial P_\beta$ according to (7.2), and therefore it holds that

$$\xi_\alpha = \xi^\gamma_\alpha \partial/\partial x^\gamma. \tag{7.4}$$

The coordinate expression of the form $d\theta$ is

$$d\theta = a^{\alpha\beta}\, dP_\alpha \wedge dP_\beta + b^\beta_\gamma\, dP_\beta \wedge dx^\gamma + c_{\gamma\delta}\, dx^\gamma \wedge dx^\delta \tag{7.5}$$

in which we suppose $a^{\alpha\beta} + a^{\beta\alpha} = 0$, $c_{\gamma\delta} + c_{\delta\gamma} = 0$. Let us consider now a linear system of equations

$$c_{\gamma\delta}\zeta^\gamma = 0 \tag{7.6}$$

for $1+n+rn-s$ unknown ζ^γ. According to (7.1), (7.5) the components ξ^γ_α of each vector field (7.4), where $\alpha = 1, 2, \ldots, s$, are the solutions of the system (7.6). By (ii) of theorem 3 the vector fields (7.4) are determined up to an arbitrary vector field η, which belongs to a certain $(1+n+rn-2s)$-dimensional distribution. Thus the system (7.6) has $s + (1+n+rn-2s) = 1+n+rn-s$ independent solutions. As the number of unknowns is equal to the number of independent solutions, the rank of the system (7.6) must be zero, i.e.

$$c_{\gamma\delta} = 0. \tag{7.7}$$

Since the form (7.5) is closed, for the remaining coefficients it holds that

$$\partial a^{\alpha\beta}/\partial P_\gamma + \partial a^{\beta\gamma}/\partial P_\alpha + \partial a^{\gamma\alpha}/\partial P_\beta = 0, \tag{7.8}$$

$$2\partial a^{\alpha\beta}/\partial x^\delta - \partial b^\alpha_\delta/\partial P_\beta + \partial b^\beta_\delta/\partial P_\alpha = 0, \tag{7.9}$$

$$\partial b^\alpha_\gamma/\partial x^\delta - \partial b^\alpha_\delta/\partial x^\gamma = 0. \tag{7.10}$$

Now let us consider an open ball $B \subset U$ with its centre at the point x_0. Then there exist two mappings $\varphi: [0, 1] \times B \to B$ and $\chi: [0, 1] \times B \to B$ defined by the relations

$$\varphi(\tau, P_1, P_2, \ldots, P_s, x^1, x^2, \ldots, x^{1+n+rn-s}) = (\tau P_1, \tau P_2, \ldots, \tau P_s, 0, 0, \ldots, 0), \tag{7.11}$$

$$\chi(\tau, P_1, P_2, \ldots, P_s, x^1, x^2, \ldots, x^{1+n+rn-s}) = (P_1, P_2, \ldots, P_s, \tau x^1, \tau x^2, \ldots, \tau x^{1+n+rn-s}). \tag{7.12}$$

On the open ball B we then introduce the functions Q^α by the quadratures

$$Q^\alpha = \int_0^1 (2\tau P_\beta \varphi^* a^{\alpha\beta} + x^\gamma \chi^* b^\alpha_\gamma)\, d\tau. \tag{7.13}$$

Then by (7.10), (7.11) and (7.12) we obtain

$$\partial Q^\alpha/\partial x^\gamma = b^\alpha_\gamma. \tag{7.14}$$

Similarly, according to (7.8), (7.9), (7.11), (7.12) and (7.13)

$$\tfrac{1}{2}(\partial Q^\alpha/\partial P_\beta - \partial Q^\beta/\partial P_\alpha) = a^{\alpha\beta}. \tag{7.15}$$

As (4.2) follows from (7.5), (7.7), (7.14) and (7.15), condition (i) of definition 4 is satisfied according to (4.7) and definition 3. By (4.4) and definition 1 the condition (ii) of definition 4 is also satisfied, so that the set Φ of functions P_α, Q^α ($\alpha = 1, 2, \ldots, s$) is a canonical set of integrals of motion. The equivalence of Φ and θ by definition 6 is apparent from the relation (4.5). This completes the proof of theorem 4.

8. Examples

8.1. Example of the canonical set associated with a given Lagrange function

Let $W = R \times R$ and let a Lagrange function be given by the relation

$$L = \tfrac{1}{2}(\dot q^2 - \ddot q^2). \tag{8.1}$$

According to (5.14) it holds that

$$\theta = (\tfrac{1}{2}\ddot q^2 - \tfrac{1}{2}\dot q^2 - \dot q \dddot q)\, dt + (\dot q + \dddot q)\, dq - \ddot q\, d\dot q, \tag{8.2}$$

so that θ is defined on j^3W.

The simplest invariant transformations of the form

$$d\theta = d(\tfrac{1}{2}\ddot q^2 - \tfrac{1}{2}\dot q^2 - \dot q \dddot q) \wedge dt + d(\dot q + \dddot q) \wedge dq + d\dot q \wedge d\ddot q \tag{8.3}$$

are translations in the variables t and q.

According to theorem 3 it follows that the expressions in the brackets are functions of the elements of the canonical set of integrals of motion. For simplicity, we choose

$$P_1 = \dot{q} + \dddot{q}, \qquad (8.4)$$
$$P_2 = \tfrac{1}{2}(\dot{q} + \dddot{q})^2 + (\tfrac{1}{2}\ddot{q}^2 - \tfrac{1}{2}\dot{q}^2 - \dot{q}\dddot{q}) = \tfrac{1}{2}\ddot{q}^2 + \tfrac{1}{2}\dddot{q}^2.$$

Moreover, we choose the set $U \subset j^3W$ so that condition (i) of definition 7 might be satisfied. As it holds that

$$\mathrm{d}P_1 \wedge \mathrm{d}P_2 = \ddot{q}(\mathrm{d}\dot{q} \wedge \mathrm{d}\ddot{q} + \mathrm{d}\dddot{q} \wedge \mathrm{d}\ddot{q}) + \dddot{q}\ \mathrm{d}\dot{q} \wedge \mathrm{d}\dddot{q} \qquad (8.5)$$

the set U can be defined by one of the relations $\ddot{q} > 0$, $\ddot{q} < 0$, $\dddot{q} > 0$, $\dddot{q} < 0$. In what follows, we shall consider only the open set $U \subset j^3W$, defined by the relation

$$\dddot{q} > 0 \qquad (8.6)$$

(if we chose another of the four possibilities, we would come to another canonical set of integrals of motion on another open subset j^3W). Since on U there are vector fields

$$\xi_1 = -\partial/\partial q, \qquad \xi_2 = -\partial/\partial t - (\dot{q} + \dddot{q})\partial/\partial q, \qquad (8.7)$$

satisfying equations (7.1), (7.2), condition (ii) of definition 7 is also satisfied, and the functions (8.4) are two independent integrals of motion in involution with respect to the Lepagean form (8.2) on the set (8.6).

As $\operatorname{rank}(\mathrm{d}\theta) = 4$, all the conditions of theorem 4 are satisfied according to definition 3, so that we can find the remaining integrals of the canonical set by means of quadratures. For that reason we exclude the coordinates $\dot{q}$ and $\dddot{q}$ from (8.3) and (8.4). Thus we obtain

$$\mathrm{d}\theta = \mathrm{d}P_1 \wedge \mathrm{d}q + \mathrm{d}P_1 \wedge \mathrm{d}\ddot{q} - P_1\,\mathrm{d}P_1 \wedge \mathrm{d}t + \mathrm{d}P_2 \wedge \mathrm{d}t - (2P_2 - \ddot{q}^2)^{-1/2}\,\mathrm{d}P_2 \wedge \mathrm{d}\ddot{q} \qquad (8.8)$$

and according to (7.5), (7.12), (7.13),

$$Q^1 = \int_0^1 (q + \ddot{q} - P_1 t)\,\mathrm{d}\tau = q + \ddot{q} - P_1 t,$$
$$Q^2 = \int_0^1 \left(t - \frac{\ddot{q}}{(2P_2 - \tau^2\ddot{q}^2)^{1/2}}\right)\mathrm{d}\tau = t - \tan^{-1}[\ddot{q}/(2P_2 - \ddot{q}^2)^{1/2}]. \qquad (8.9)$$

From (8.4) and (8.9) it follows that the canonical set of integrals of motion on the set (8.6), equivalent to the Lepagean form (8.2), is formed by the functions

$$P_1 = \dot{q} + \dddot{q}, \qquad P_2 = \tfrac{1}{2}\ddot{q}^2 + \tfrac{1}{2}\dddot{q}^2,$$
$$Q^1 = q + \ddot{q} - (\dot{q} + \dddot{q})t, \qquad Q^2 = t - \tan^{-1}(\ddot{q}/\dddot{q}). \qquad (8.10)$$

We note that higher-order mechanical systems correspond generally within the frame of the Newton approach, to systems with the 'hidden motions'. In our case the Lagrange function (8.1) determines the one-dimensional motion of a 'black box' in which the harmonic oscillator is hidden. A system of units is chosen in such a way that the angular frequency of oscillations is unity. The physical meaning of the integrals of motion P_1, P_2, Q^1, Q^2 is as follows: the momentum of the centre of mass, the energy of oscillations, the starting position of the centre of mass and the starting phase of oscillations.

8.2. Example of the Lagrange function, associated with given integrals of motion

The question we discuss in this section by means of the canonical set of integrals of motion has been already solved, in a different way, by Darboux (1894). We set $W = R \times R$ and consider the equation of motion of the system in the form

$$j^2\gamma^*(\ddot{q} - f(q, \dot{q}, t)) = 0, \tag{8.11}$$

where $f(q, \dot{q}, t)$ is a given function and γ is a critical section of the fibred manifold π. In such a case every function F on j^1W satisfying the condition

$$\partial F/\partial t + (\partial F/\partial q)\dot{q} + (\partial F/\partial \dot{q})f = 0 \tag{8.12}$$

is an integral of motion. This equation can be rewritten by means of the total derivative in the form

$$\mathrm{d}F/\mathrm{d}t = (\partial F/\partial \dot{q})(\ddot{q} - f). \tag{8.13}$$

Consider, now, any two integrals of motion P, Q on an open set $U \subset j^1W$ such that at each point of the set U it holds that

$$(\partial P/\partial \dot{q})\partial Q/\partial q - (\partial P/\partial q)\partial Q/\partial \dot{q} \neq 0. \tag{8.14}$$

Then for $s = 1$ condition (i) of definition 4 is satisfied. As for as condition (ii) of definition 4 is concerned it can be transformed to the equivalent form

$$(\partial P/\partial \dot{q})\mathrm{d}Q/\mathrm{d}t - (\partial Q/\partial \dot{q})\mathrm{d}P/\mathrm{d}t = 0, \tag{8.15}$$

in which it is, by (8.13), satisfied as well. In our case any two independent integrals of motion form a canonical set Φ of the integrals of motion.

The sought after Lagrange function is then, according to (5.15), given by the relation

$$L = P\,\mathrm{d}Q/\mathrm{d}t + \mathrm{d}S/\mathrm{d}t \tag{8.16}$$

where S is an arbitrary function on j^1W. To find out that the corresponding Euler–Lagrange equations (5.2) are equivalent to (8.11) we shall rewrite the expression (5.3) by means of (8.16) in the form

$$\varepsilon = \left(\frac{\partial P}{\partial q}\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\partial Q}{\partial q}\frac{\mathrm{d}P}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial Q}{\partial \dot{q}}\frac{\mathrm{d}P}{\mathrm{d}t} - \frac{\partial P}{\partial \dot{q}}\frac{\mathrm{d}Q}{\mathrm{d}t}\right). \tag{8.17}$$

By using (8.13) for the total derivatives we get the relation

$$\varepsilon = [(\partial P/\partial \dot{q})\partial Q/\partial q - (\partial P/\partial q)\partial Q/\partial \dot{q}](f - \ddot{q}) \tag{8.18}$$

from which, by (8.14), the equivalence is apparent.

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